Analysis of Log-Weighted Quadrature Domains

and the Faber Transform

Andrew Graven
Caltech Department of Mathematics

UT Dallas Complex (Analysis and Dynamics) Conference II
March 29, 2026

Table of Contents

Quadrature Identities

Mean value property:[1]

$$f \in \mathcal{A}(\mathbb{D}_r(w_0)) \quad \Longrightarrow \quad \frac{1}{r^2}\int_{\mathbb{D}_r(w_0)} f dA = f(w_0).$$

Epstein & Schiffer (1965): $\mathbb{D}_r(w_0)$ is the only[2] domain with this property.

[1] $dA = \frac{dxdy}{\pi}$

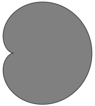
[2] bounded & simply connected

Quadrature Identities

Mean value property:[1]

$$f \in \mathcal{A}(\mathbb{D}_r(w_0)) \quad \Longrightarrow \quad \frac{1}{r^2}\int_{\mathbb{D}_r(w_0)} f dA = f(w_0).$$

Epstein & Schiffer (1965): $\mathbb{D}_r(w_0)$ is the only[2] domain with this property.

The cardioid, $\Omega = \left\{z + \frac{z^2}{2} : z \in \mathbb{D}\right\}$: 

$$f \in \mathcal{A}(\Omega) \quad \Longrightarrow \quad \int_\Omega f dA = \frac{3}{2} f(0) + \frac{1}{2} f'(0).$$

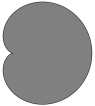
[1] $dA = \frac{dxdy}{\pi}$
[2] bounded & simply connected

Quadrature Identities

Mean value property:[1]

$$f \in \mathcal{A}(\mathbb{D}_r(w_0)) \implies \frac{1}{r^2}\int_{\mathbb{D}_r(w_0)} f dA = f(w_0).$$

Epstein & Schiffer (1965): $\mathbb{D}_r(w_0)$ is the only[2] domain with this property.

The cardioid, $\Omega = \left\{z + \frac{z^2}{2} : z \in \mathbb{D}\right\}$: 

$$f \in \mathcal{A}(\Omega) \implies \int_\Omega f dA = \frac{3}{2}f(0) + \frac{1}{2}f'(0).$$

Aharonov & Shapiro (1976): The cardioid is also unique.

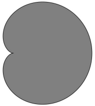
[1] $dA = \frac{dxdy}{\pi}$
[2] bounded & simply connected

Quadrature Identities

Mean value property:[1]

$$f \in \mathcal{A}(\mathbb{D}_r(w_0)) \implies \frac{1}{r^2}\int_{\mathbb{D}_r(w_0)} f dA = f(w_0).$$

Epstein & Schiffer (1965): $\mathbb{D}_r(w_0)$ is the only[2] domain with this property.

The cardioid, $\Omega = \left\{ z + \frac{z^2}{2} : z \in \mathbb{D} \right\}$: 

$$f \in \mathcal{A}(\Omega) \implies \int_\Omega f dA = \frac{3}{2} f(0) + \frac{1}{2} f'(0).$$

Aharonov & Shapiro (1976): The cardioid is also unique.

These are examples of *quadrature identities*.

[1] $dA = \frac{dxdy}{\pi}$

[2] bounded & simply connected

Bounded Quadrature Domains

Definition 1.1 (Bounded Quadrature domain)

A bounded domain $\Omega \subset \widehat{\mathbb{C}}$ is a *bounded* QD if there exists rational h s.t.

$$\int_{\Omega} f dA = \oint_{\partial\Omega} f(w)h(w)dw$$

$\forall f \in \mathcal{A}(\Omega)$. This is denoted by $\Omega \in \mathrm{QD}(h)$.

Bounded Quadrature Domains

Definition 1.1 (Bounded Quadrature domain)

A bounded domain $\Omega \subset \widehat{\mathbb{C}}$ is a *bounded* QD if there exists rational h s.t.

$$\int_{\Omega} f dA = \oint_{\partial\Omega} f(w)h(w)dw$$

$\forall f \in \mathcal{A}(\Omega)$. This is denoted by $\Omega \in \mathrm{QD}(h)$.

This is equivalent to the existence of a quadrature identity:

$$\oint_{\partial\Omega} f(w)h(w)dw = \sum_{p_k \in h^{-1}(\infty)} \operatorname*{Res}_{w=p_k} (f(w)h(w)) = \sum_{k,j} c_{k,j} f^{(n_j)}(p_k).$$

Definition 1.2 (Unbounded Quadrature Domain)

An unbounded domain $\Omega \subset \widehat{\mathbb{C}}$ is an *unbounded* QD if $\exists$ a rational h s.t.

$$\int_\Omega f dA = \oint_{\partial\Omega} f(w)h(w)dw$$

$\forall f \in \mathcal{A}_0(\Omega)$. This is denoted by $\Omega \in \mathrm{QD}(h)$.

Definition 1.2 (Unbounded Quadrature Domain)

An unbounded domain $\Omega \subset \widehat{\mathbb{C}}$ is an *unbounded* QD if $\exists$ a rational h s.t.

$$\int_{\Omega} f dA = \oint_{\partial\Omega} f(w)h(w)dw$$

$\forall f \in \mathcal{A}_0(\Omega)$. This is denoted by $\Omega \in \mathrm{QD}(h)$.

unbounded QD $\longleftrightarrow$ quadrature identity

$$\oint_{\partial\Omega} f(w)h(w)dw = \sum_{k,j} c_{k,j} f^{(n_j)}(p_k) + \sum_{j} c_j f_j$$

where $f(w) = \sum_{j=1}^{\infty} f_j w^{-j}$.

Characterizations of QDs I: "Potato Kugel"[3]

Definition 1.3 (Cauchy transform)

For a Borel set $\Omega \subset \mathbb{C}$, we denote the *Cauchy transform* of Ω by $C^{\Omega} : \mathbb{C} \to \mathbb{C}$,

$$C^{\Omega}(w) = \int_{\Omega} \frac{dA(\xi)}{w - \xi}$$

$$\Omega \in \mathrm{QD}(h) \iff C^{\Omega} = h \text{ in } \Omega_{*}$$

[3] Aharonov, Schiffer & Zalcman (1981)

Characterizations of QDs I: "Potato Kugel"[3]

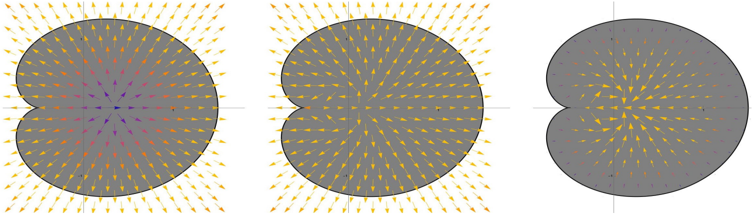
Definition 1.3 (Cauchy transform)

For a Borel set $\Omega \subset \mathbb{C}$, we denote the *Cauchy transform* of Ω by $C^\Omega : \mathbb{C} \to \mathbb{C}$,

$$C^\Omega(w) = \int_\Omega \frac{dA(\xi)}{w - \xi}$$

$$\Omega \in \mathrm{QD}(h) \quad \Longleftrightarrow \quad C^\Omega = h \text{ in } \Omega_*$$

$\overline{C^\Omega}$ corresponds to the electric field due to a uniform charge distribution on Ω.



$\overline{C^\Omega}$ $\qquad$ $\overline{h}$ $\qquad$ $\overline{C^\Omega} - \overline{h}$

[3] Aharonov, Schiffer & Zalcman (1981)

Characterizations of QDs II: The Coincidence Equation

$\Omega \subset \widehat{\mathbb{C}}$ is a QD iff there exists a rational function h such that

$$\overline{w} = h(w) + C^{\Omega^c}(w), \quad \forall w \in \partial\Omega.$$

Referred to as the *coincidence equation*.

Characterizations of QDs II: The Coincidence Equation

$\Omega \subset \widehat{\mathbb{C}}$ is a QD iff there exists a rational function h such that

$$\overline{w} = h(w) + C^{\Omega^c}(w), \quad \forall w \in \partial\Omega.$$

Referred to as the *coincidence equation*.

We interpret this identity in terms of logarithmic potential theory: If $K = \Omega^c$ is a local droplet of the potential

$$Q(w) = |w|^2 - 2\mathrm{Re}(H(w)),$$

then

$$\frac{\partial}{\partial \overline{w}}\left(Q(w) + U^{\mu}(w)\right) = 0, \quad \forall w \in K.$$

If we set $h = H'$ then differentiating, yields the coincidence equation.

Characterizations of QDs III: The Schwarz Function

$\Omega \subset \widehat{\mathbb{C}}$ is a QD $\iff$ Ω admits a *Schwarz function*.

Characterizations of QDs III: The Schwarz Function

$\Omega \subset \widehat{\mathbb{C}}$ is a QD $\iff$ Ω admits a *Schwarz function*.

S is called a *Schwarz function* for Ω if:

1. $S \in \mathcal{M}(\Omega)$,
2. S extends continuously to $\partial\Omega$,
3. $S(w) = \overline{w}$ on $\partial\Omega$.

Characterizations of QDs III: The Schwarz Function

$$\Omega \subset \widehat{\mathbb{C}} \text{ is a QD} \iff \Omega \text{ admits a } \textit{Schwarz function}.$$

S is called a *Schwarz function* for Ω if:

1. $S \in \mathcal{M}(\Omega)$,
2. S extends continuously to $\partial\Omega$,
3. $S(w) = \overline{w}$ on $\partial\Omega$.

S can be represented explicitly in terms of h and the Cauchy transform:

$$S(w) = h(w) + C^{\Omega^c}(w).$$

Characterizations of QDs IV: The Riemann Map

If Ω is simply connected, then $\Omega \in \mathrm{QD}$ iff its Riemann map is rational.

Characterizations of QDs IV: The Riemann Map

If Ω is simply connected, then $\Omega \in \mathrm{QD}$ iff its Riemann map is rational.

Moreover, if $\Omega \in \mathrm{QD}(h)$ is a **bounded** domain with Riemann map φ, then

$$h(w) = \Phi_\varphi\left(\varphi^\# - \overline{\varphi(0)}\right)(w)$$

$$\varphi(z) = \varphi(0) + \Phi_\varphi^{-1}(h)^\#(z)$$

Characterizations of QDs IV: The Riemann Map

If Ω is simply connected, then $\Omega \in \mathrm{QD}$ iff its Riemann map is rational.

Moreover, if $\Omega \in \mathrm{QD}(h)$ is a **bounded** domain with Riemann map φ, then

$$h(w) = \Phi_\varphi \left(\varphi^\# - \overline{\varphi(0)} \right)(w)$$

$$\varphi(z) = \varphi(0) + \Phi_\varphi^{-1}(h)^\#(z)$$

If Ω is **unbounded**, then

$$h(w) = \Phi_\varphi \left(\varphi^\# - cz^{-1} \right)(w)$$

$$\varphi(z) = cz + \Phi_\varphi^{-1}(h)^\#(z)$$

Where Φ_φ is the *Faber transform*.

Summary of QD Equivalent Characterizations

The following are equivalent to $\Omega \in \mathrm{QD}(h)$:

1. $C^{\Omega} = h$ in Ω_*;
2. Ω satisfies the coincidence equation;
3. Ω admits a Schwarz function;
4. Ω has a rational Riemann map (when simply connected).

The Faber Transform

Let $\Omega \subset \widehat{\mathbb{C}}$ be unbounded and simply connected with Riemann map $\varphi : \mathbb{D}_* \to \Omega$,

$$\varphi(z) = cz + f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots , \qquad c = \mathrm{rad}_\infty(\Omega) > 0$$

The Faber Transform

Let $\Omega \subset \widehat{\mathbb{C}}$ be unbounded and simply connected with Riemann map $\varphi : \mathbb{D}_* \to \Omega$,

$$\varphi(z) = cz + f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots , \qquad c = \mathrm{rad}_\infty(\Omega) > 0$$

The associated *exterior Faber transform* Φ_φ is a linear iso $\mathcal{A}(\mathbb{D}) \to \mathcal{A}(\Omega_*)$,

$$\Phi_\varphi(f)(w) = \oint_{\partial\mathbb{D}_*} \frac{f(z)\varphi'(z)}{\varphi(z) - w} dz$$

The Faber Transform

Let $\Omega \subset \widehat{\mathbb{C}}$ be unbounded and simply connected with Riemann map $\varphi : \mathbb{D}_* \to \Omega$,

$$\varphi(z) = cz + f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots , \qquad c = \operatorname{rad}_\infty(\Omega) > 0$$

The associated *exterior Faber transform* Φ_φ is a linear iso $\mathcal{A}(\mathbb{D}) \to \mathcal{A}(\Omega_*)$,

$$\Phi_\varphi(f)(w) = \oint_{\partial\mathbb{D}_*} \frac{f(z)\varphi'(z)}{\varphi(z) - w} dz$$

Note: the Faber transform preserves polynomials and rational functions, e.g.

$$\Phi_\varphi\left(\frac{1}{z - z_0}\right)(w) = \frac{\varphi'(z_0)}{w - \varphi(z_0)},$$

$$\Phi_\varphi(z^n)(w) =: F_n(w) \quad (n\text{th Faber polynomial})$$

The Faber Transform

Let $\Omega \subset \widehat{\mathbb{C}}$ be unbounded and simply connected with Riemann map $\varphi : \mathbb{D}_* \to \Omega$,

$$\varphi(z) = cz + f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots , \qquad c = \mathrm{rad}_\infty(\Omega) > 0$$

The associated *exterior Faber transform* Φ_φ is a linear iso $\mathcal{A}(\mathbb{D}) \to \mathcal{A}(\Omega_*)$,

$$\Phi_\varphi(f)(w) = \oint_{\partial \mathbb{D}_*} \frac{f(z)\varphi'(z)}{\varphi(z) - w} dz$$

Note: the Faber transform preserves polynomials and rational functions, e.g.

$$\Phi_\varphi \left(\frac{1}{z - z_0} \right) (w) = \frac{\varphi'(z_0)}{w - \varphi(z_0)},$$

$$\Phi_\varphi (z^n) (w) =: F_n(w) \ \ (n\text{th Faber polynomial})$$

The *interior* Faber transform, defined for bounded s.c. domains is defined similarly, and is a map $\Phi_\varphi : \mathcal{A}_0(\mathbb{D}_*) \to \mathcal{A}_0(\Omega_*)$.

Log-Weighted Quadrature Domains

Definition 2.1 (Log-Weighted Quadrature Domain)

Let $\Omega \subset \widehat{\mathbb{C}}$ be a domain for which $0, \infty \notin \partial\Omega$. We say that Ω is a log-weighted quadrature domain (LQD) if there exists a rational h such that

$$\int_\Omega f(w)|w|^{-2} dA(w) = \oint_{\partial\Omega} f(w)h(w)dw, \qquad \forall f \in L_a^1(\Omega; \rho_0).$$

Denoted by $\Omega \in \mathrm{QD}_0(h)$.

That is, LQDs are quadrature domains with respect to the weight $\rho_0(w) = |w|^{-2}$.

Log-Weighted Quadrature Domains

Definition 2.1 (Log-Weighted Quadrature Domain)

Let $\Omega \subset \widehat{\mathbb{C}}$ be a domain for which $0, \infty \notin \partial\Omega$. We say that Ω is a log-weighted quadrature domain (LQD) if there exists a rational h such that

$$\int_\Omega f(w)|w|^{-2} dA(w) = \oint_{\partial\Omega} f(w)h(w)dw, \qquad \forall f \in L^1_a(\Omega; \rho_0).$$

Denoted by $\Omega \in \mathrm{QD}_0(h)$.

That is, LQDs are quadrature domains with respect to the weight $\rho_0(w) = |w|^{-2}$.

- The primary distinction of LQDs from classical QDs is the metric singularity at $w = 0$, reflected in the constraint $f \in L^1_a(\Omega; \rho_0) \Rightarrow f(0) = 0$.

Log-Weighted Quadrature Domains

Definition 2.1 (Log-Weighted Quadrature Domain)

Let $\Omega \subset \widehat{\mathbb{C}}$ be a domain for which $0, \infty \notin \partial\Omega$. We say that Ω is a log-weighted quadrature domain (LQD) if there exists a rational h such that

$$\int_\Omega f(w)|w|^{-2}dA(w) = \oint_{\partial\Omega} f(w)h(w)dw, \qquad \forall f \in L^1_a(\Omega;\rho_0).$$

Denoted by $\Omega \in \mathrm{QD}_0(h)$.

That is, LQDs are quadrature domains with respect to the weight $\rho_0(w) = |w|^{-2}$.

- The primary distinction of LQDs from classical QDs is the metric singularity at $w = 0$, reflected in the constraint $f \in L^1_a(\Omega;\rho_0) \Rightarrow f(0) = 0$.
- We refer to LQDs containing zero as *singular*.

Log-Weighted Quadrature Domains

Definition 2.1 (Log-Weighted Quadrature Domain)

Let $\Omega \subset \widehat{\mathbb{C}}$ be a domain for which $0, \infty \notin \partial\Omega$. We say that Ω is a log-weighted quadrature domain (LQD) if there exists a rational h such that

$$\int_{\Omega} f(w)|w|^{-2} dA(w) = \oint_{\partial\Omega} f(w)h(w)dw, \qquad \forall f \in L^1_a(\Omega; \rho_0).$$

Denoted by $\Omega \in \mathrm{QD}_0(h)$.

That is, LQDs are quadrature domains with respect to the weight $\rho_0(w) = |w|^{-2}$.

- The primary distinction of LQDs from classical QDs is the metric singularity at $w = 0$, reflected in the constraint $f \in L^1_a(\Omega; \rho_0) \Rightarrow f(0) = 0$.
- We refer to LQDs containing zero as *singular*.
- The singularity plays a similar role to $\infty \in \widehat{\mathbb{C}}$ for classical QDs.

Log-Weighted Quadrature Domains

Definition 2.1 (Log-Weighted Quadrature Domain)

Let $\Omega \subset \widehat{\mathbb{C}}$ be a domain for which $0, \infty \notin \partial\Omega$. We say that Ω is a log-weighted quadrature domain (LQD) if there exists a rational h such that

$$\int_{\Omega} f(w)|w|^{-2} dA(w) = \oint_{\partial\Omega} f(w)h(w)dw, \qquad \forall f \in L^1_a(\Omega; \rho_0).$$

Denoted by $\Omega \in \mathrm{QD}_0(h)$.

That is, LQDs are quadrature domains with respect to the weight $\rho_0(w) = |w|^{-2}$.

- The primary distinction of LQDs from classical QDs is the metric singularity at $w = 0$, reflected in the constraint $f \in L^1_a(\Omega; \rho_0) \Rightarrow f(0) = 0$.
- We refer to LQDs containing zero as *singular*.
- The singularity plays a similar role to $\infty \in \widehat{\mathbb{C}}$ for classical QDs.
- h is *not* always unique: If $0 \in \Omega \in \mathrm{QD}_0(h)$, then $\Omega \in \mathrm{QD}_0\left(h(w) + \frac{q}{w}\right)$ for all $q \in \mathbb{C}$.

A First Example

Fact: If Ω is a bounded domain, then

$$\Omega \in \mathrm{QD}(h) \implies e^{\Omega} \in \mathrm{QD}_0(\widetilde{h}),$$

for some rational $\widetilde{h}$. Moreover, if $\{p_k\}$ are the poles of h then $\{e^{p_k}\}$ are the poles of $\widetilde{h}$, with the same multiplicity.

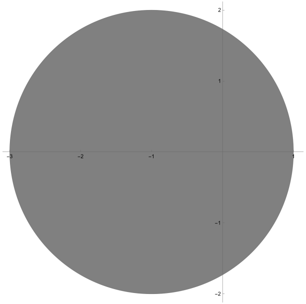
A First Example

Fact: If Ω is a bounded domain, then

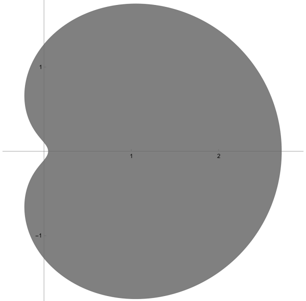
$$\Omega \in \mathrm{QD}(h) \quad \Longrightarrow \quad e^{\Omega} \in \mathrm{QD}_0(\widetilde{h}),$$

for some rational $\widetilde{h}$. Moreover, if $\{p_k\}$ are the poles of h then $\{e^{p_k}\}$ are the poles of $\widetilde{h}$, with the same multiplicity.

$$\mathbb{D}_2(-1) \in \mathrm{QD}\left(\frac{4}{w+1}\right) \qquad \Longrightarrow \qquad e^{\mathbb{D}_2(-1)} \in \mathrm{QD}_0\left(\frac{4}{w-e^{-1}}\right)$$



$$\xrightarrow{w \;\mapsto\; e^{w}}$$



Characterizations of LQDs I: Potato Kugel

Definition 2.2 (Weighted Cauchy transform)

For a Borel set $\Omega \subset \mathbb{C}$, we denote the ρ_0-*weighted Cauchy transform* of Ω by $C^{\Omega}_{\rho_0} : \mathbb{C} \to \mathbb{C}$,

$$C^{\Omega}_{\rho_0}(w) = \int_{\Omega} \frac{|\xi|^{-2} dA(\xi)}{w - \xi}$$

$$\Omega \in \mathrm{QD}_0 \iff C^{\Omega}_{\rho_0} \text{ extends to a rational function.}$$

$\overline{C^{\Omega}_{\rho_0}}$ corresponds to the electric field due to a charge density $\rho_0(w) = |w|^{-2}$ on Ω:

Caltech

Characterizations of LQDs I: Potato Kugel

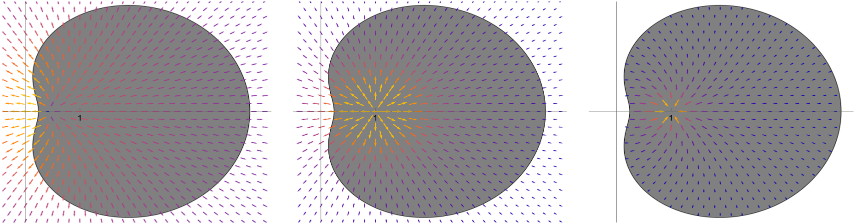
Definition 2.2 (Weighted Cauchy transform)

For a Borel set $\Omega \subset \mathbb{C}$, we denote the ρ_0-*weighted Cauchy transform* of Ω by $C^{\Omega}_{\rho_0} : \mathbb{C} \to \mathbb{C}$,

$$C^{\Omega}_{\rho_0}(w) = \int_{\Omega} \frac{|\xi|^{-2} dA(\xi)}{w - \xi}$$

$$\Omega \in \mathrm{QD}_0 \iff C^{\Omega}_{\rho_0} \text{ extends to a rational function.}$$

$\overline{C^{\Omega}_{\rho_0}}$ corresponds to the electric field due to a charge density $\rho_0(w) = |w|^{-2}$ on Ω:



$\overline{C^{\Omega}_{\rho_0}}$ $\quad$ $\overline{h}$ $\quad$ $\overline{C^{\Omega}_{\rho_0}} - \overline{h}$

Characterizations of LQDs II: The Coincidence Equation

Non-singular coincidence equation: A domain $0 \notin \Omega$ is an LQD iff there exist $G \in \mathcal{A}(\Omega)$ and a rational function h such that

$$\frac{\ln |w|^2}{w} = h(w) + G(w), \quad \forall w \in \partial\Omega.$$

Characterizations of LQDs II: The Coincidence Equation

Non-singular coincidence equation: A domain $0 \notin \Omega$ is an LQD iff there exist $G \in \mathcal{A}(\Omega)$ and a rational function h such that

$$\frac{\ln |w|^2}{w} = h(w) + G(w), \quad \forall w \in \partial\Omega.$$

Singular coincidence equation: A domain $0 \in \Omega$ is an LQD iff there exist $q \in \mathbb{C}$, $G \in \mathcal{A}(\Omega)$, and a rational function h such that

$$\frac{\ln |w|^2}{w} = h(w) + \frac{q}{w} + G(w), \quad \forall w \in \partial\Omega.$$

In both cases, $\Omega \in \mathrm{QD}_0(h)$.

Characterizations of LQDs III: The Generalized S-Function

Theorem 2.3

A domain Ω is an LQD iff it admits a generalized Schwarz function.

Characterizations of LQDs III: The Generalized S-Function

Theorem 2.3

A domain Ω is an LQD iff it admits a generalized Schwarz function.

S_0 is called a *generalized Schwarz function* for Ω if

1. $S_0 \in \mathcal{M}(\Omega)$,
2. S_0 extends continuously to $\partial\Omega$,
3. $S_0(w) = \frac{\ln |w|^2}{w}$ on $\partial\Omega$.

Characterizations of LQDs III: The Generalized S-Function

Theorem 2.3

A domain Ω is an LQD iff it admits a generalized Schwarz function.

S_0 is called a *generalized Schwarz function* for Ω if

1. $S_0 \in \mathcal{M}(\Omega)$,
2. S_0 extends continuously to $\partial\Omega$,
3. $S_0(w) = \frac{\ln |w|^2}{w}$ on $\partial\Omega$.

Remark: Solving for $\overline{w}$, we recover an analogue of the classical Schwarz function $S|_{\partial\Omega}(w) = \overline{w}$:

$$S(w) := \frac{1}{w} e^{wS_0(w)}.$$

Characterizations of LQDs III: The Generalized S-Function

Theorem 2.3

A domain Ω is an LQD iff it admits a generalized Schwarz function.

S_0 is called a *generalized Schwarz function* for Ω if

1. $S_0 \in \mathcal{M}(\Omega)$,
2. S_0 extends continuously to $\partial\Omega$,
3. $S_0(w) = \frac{\ln|w|^2}{w}$ on $\partial\Omega$.

Remark: Solving for $\overline{w}$, we recover an analogue of the classical Schwarz function $S|_{\partial\Omega}(w) = \overline{w}$:

$$S(w) := \frac{1}{w}e^{wS_0(w)}.$$

Not typically a Schwarz function (essential singularity $\implies$ not meromorphic).

Characterizations of LQDs IV: The Riemann Map

Let Ω be a simply connected domain with Riemann map φ.

Write the *inner/outer factorization* of φ:

$$\varphi(z) = \varphi_{\mathrm{in}}(z)\varphi_{\mathrm{out}}(z).$$

Characterizations of LQDs IV: The Riemann Map

Let Ω be a simply connected domain with Riemann map φ.

Write the *inner/outer factorization* of φ:

$$\varphi(z) = \varphi_{\mathrm{in}}(z)\varphi_{\mathrm{out}}(z).$$

Theorem 2.4

$\Omega \in QD_0$ *iff* $\ln(\varphi_{\mathrm{out}})$ *extends to a rational function.*

Characterizations of LQDs IV: The Riemann Map

Let Ω be a simply connected domain with Riemann map φ.

Write the *inner/outer factorization* of φ:

$$\varphi(z) = \varphi_{\text{in}}(z)\varphi_{\text{out}}(z).$$

Theorem 2.4

$\Omega \in QD_0$ *iff* $\ln(\varphi_{\text{out}})$ *extends to a rational function.*

$\varphi(z) = \varphi_{\text{in}}(z)\varphi_{\text{out}}(z)$ in the sense of Nevanlinna theory.

In general:

- φ_{in} is a ratio of Blaschke factors:

$$b_{z_0}(z) := \frac{\overline{z_0}}{|z_0|}\frac{z - z_0}{\overline{z_0}z - 1},$$

- φ_{out} is non-zero and analytic.

Characterizations of LQDs IV: The Riemann Map

If Ω is a *bounded* domain, then we can write the inner/outer factorization of φ as follows:

$$\begin{aligned} \varphi(z) &= 1 \cdot \varphi_{\mathrm{out}}(z), && \text{when } 0 \notin \Omega, \\ \varphi(z) &= b_{z_0}(z) \cdot \varphi_{\mathrm{out}}(z), && \text{when } 0 \in \Omega, \end{aligned}$$

where z_0 is the unique root of φ in $\mathbb{D}$.

Characterizations of LQDs IV: The Riemann Map

If Ω is a *bounded* domain, then we can write the inner/outer factorization of φ as follows:

$$\varphi(z) = 1 \cdot \varphi_{\mathrm{out}}(z), \qquad \text{when } 0 \notin \Omega,$$
$$\varphi(z) = b_{z_0}(z) \cdot \varphi_{\mathrm{out}}(z), \qquad \text{when } 0 \in \Omega,$$

where z_0 is the unique root of φ in $\mathbb{D}$.

If Ω is an *unbounded* domain, then

$$\varphi(z) = z \cdot \varphi_{\mathrm{out}}(z), \qquad \text{when } 0 \notin \Omega,$$
$$\varphi(z) = z b_{z_0}(z) \cdot \varphi_{\mathrm{out}}(z), \qquad \text{when } 0 \in \Omega,$$

where z_0 is the unique root of φ in $\mathbb{D}_*$.

Characterizations of LQDs IV: The Riemann Map

If Ω is a *bounded* domain, then we can write the inner/outer factorization of φ as follows:

$$\begin{aligned} \varphi(z) &= 1 \cdot \varphi_{\mathrm{out}}(z), && \text{when } 0 \notin \Omega, \\ \varphi(z) &= b_{z_0}(z) \cdot \varphi_{\mathrm{out}}(z), && \text{when } 0 \in \Omega, \end{aligned}$$

where z_0 is the unique root of φ in $\mathbb{D}$.

If Ω is an *unbounded* domain, then

$$\begin{aligned} \varphi(z) &= z \cdot \varphi_{\mathrm{out}}(z), && \text{when } 0 \notin \Omega, \\ \varphi(z) &= z b_{z_0}(z) \cdot \varphi_{\mathrm{out}}(z), && \text{when } 0 \in \Omega, \end{aligned}$$

where z_0 is the unique root of φ in $\mathbb{D}_*$.

Note: This factorization is key to relating the Riemann map and quadrature function.

Summary of LQD Equivalent Characterizations

The following are equivalent to $\Omega \in \mathrm{QD}_0(h)$:

1. The weighted Cauchy transform:
   - $0 \notin \Omega$: $C_{\rho_0}^{\Omega} = h$,
   - $0 \notin \Omega$: $C_{\rho_0}^{\Omega_0 \setminus \mathbb{D}_r}(w) = \dfrac{q}{w} + h(w)$.

Summary of LQD Equivalent Characterizations

The following are equivalent to $\Omega \in \mathrm{QD}_0(h)$:

1. The weighted Cauchy transform:
   - $0 \notin \Omega$: $C_{\rho_0}^{\Omega} = h$,
   - $0 \notin \Omega$: $C_{\rho_0}^{\Omega_0 \setminus \mathbb{D}_r}(w) = \dfrac{q}{w} + h(w)$.
2. Ω satisfies the generalized coincidence equation:

$$\frac{\ln |w|^2}{w} = h(w) + \frac{q}{w} + G(w), \quad \forall w \in \partial\Omega.$$

Summary of LQD Equivalent Characterizations

The following are equivalent to $\Omega \in \mathrm{QD}_0(h)$:

1. The weighted Cauchy transform:
   - $0 \notin \Omega$: $C^{\Omega}_{\rho_0} = h$,
   - $0 \notin \Omega$: $C^{\Omega_0 \setminus \mathbb{D}_r}_{\rho_0}(w) = \dfrac{q}{w} + h(w)$.
2. Ω satisfies the generalized coincidence equation:

$$\frac{\ln |w|^2}{w} = h(w) + \frac{q}{w} + G(w), \quad \forall w \in \partial\Omega.$$

3. Ω admits a generalized Schwarz function:

$$S_0(w) = \frac{\ln |w|^2}{w}, \quad \forall w \in \partial\Omega.$$

Summary of LQD Equivalent Characterizations

The following are equivalent to $\Omega \in \mathrm{QD}_0(h)$:

1. The weighted Cauchy transform:
   - $0 \notin \Omega$: $C^{\Omega}_{\rho_0} = h$,
   - $0 \notin \Omega$: $C^{\Omega_0 \setminus \mathbb{D}_r}_{\rho_0}(w) = \frac{q}{w} + h(w)$.
2. Ω satisfies the generalized coincidence equation:

$$\frac{\ln |w|^2}{w} = h(w) + \frac{q}{w} + G(w), \quad \forall w \in \partial\Omega.$$

3. Ω admits a generalized Schwarz function:

$$S_0(w) = \frac{\ln |w|^2}{w}, \quad \forall w \in \partial\Omega.$$

4. $\ln(\varphi_{\mathrm{out}})$ extends to a rational function (if simply connected).

Other Properties

Theorem 2.5 (Boundary regularity)

If $\Omega \in QD_0$ then $\partial\Omega$ has finitely many singular points, and each is a cusp or a double point.

Other Properties

Theorem 2.5 (Boundary regularity)

If $\Omega \in QD_0$ then $\partial\Omega$ has finitely many singular points, and each is a cusp or a double point.

Theorem 2.6 (Scaling law)

For each $a \in \mathbb{C} \setminus \{0\}$, $\Omega \in QD_0(h)$ if and only if $a^{-1}\Omega \in QD_0(h(aw)a)$.

Other Properties

Theorem 2.5 (Boundary regularity)

If $\Omega \in QD_0$ then $\partial\Omega$ has finitely many singular points, and each is a cusp or a double point.

Theorem 2.6 (Scaling law)

For each $a \in \mathbb{C} \setminus \{0\}$, $\Omega \in QD_0(h)$ if and only if $a^{-1}\Omega \in QD_0(h(aw)a)$.

Theorem 2.7 (Invariance under inversion)

If $\Omega \subseteq \widehat{\mathbb{C}}$ then $\Omega \in QD_0(h)$ iff $\Omega^{-1} \in QD_0(-h(w^{-1})w^{-2})$.

(the singularities at $w = 0$ and $w = \infty$ are "interchangable")

The Inverse and Direct Problems for LQDs

Question: When is an LQD uniquely associated to its quadrature function? How can we recover it?

These are open problems even for *classical* QDs.

[4]Novikoff, P.S.: Sur le problème inverse du potentiel. (1938)

[5]Gustafsson, B., Putinar, M.: Selected topics on quadrature domains. (2007)

[6]Aharonov, D., Shapiro, H.S.: Domains on which analytic functions satisfy quadrature identities. (1976)

The Inverse and Direct Problems for LQDs

Question: When is an LQD uniquely associated to its quadrature function? How can we recover it?

These are open problems even for *classical* QDs.

Some known results for QDs:

[4] Novikoff, P.S.: Sur le problème inverse du potentiel. (1938)

[5] Gustafsson, B., Putinar, M.: Selected topics on quadrature domains. (2007)

[6] Aharonov, D., Shapiro, H.S.: Domains on which analytic functions satisfy quadrature identities. (1976)

The Inverse and Direct Problems for LQDs

Question: When is an LQD uniquely associated to its quadrature function? How can we recover it?

These are open problems even for *classical* QDs.

Some known results for QDs:

- The disk is the unique finitely connected *bounded* domain with the MVP.

[4] Novikoff, P.S.: Sur le problème inverse du potentiel. (1938)
[5] Gustafsson, B., Putinar, M.: Selected topics on quadrature domains. (2007)
[6] Aharonov, D., Shapiro, H.S.: Domains on which analytic functions satisfy quadrature identities. (1976)

The Inverse and Direct Problems for LQDs

Question: When is an LQD uniquely associated to its quadrature function? How can we recover it?

These are open problems even for *classical* QDs.

Some known results for QDs:

- The disk is the unique finitely connected *bounded* domain with the MVP.
- $\Omega \in QD(0)$ (Ω is a "null" QD) iff Ω is an exterior disk

[4] Novikoff, P.S.: Sur le problème inverse du potentiel. (1938)
[5] Gustafsson, B., Putinar, M.: Selected topics on quadrature domains. (2007)
[6] Aharonov, D., Shapiro, H.S.: Domains on which analytic functions satisfy quadrature identities. (1976)

The Inverse and Direct Problems for LQDs

Question: When is an LQD uniquely associated to its quadrature function? How can we recover it?

These are open problems even for *classical* QDs.

Some known results for QDs:

- The disk is the unique finitely connected *bounded* domain with the MVP.
- $\Omega \in QD(0)$ (Ω is a "null" QD) iff Ω is an exterior disk
- If $\Omega_1, \Omega_2 \in QD(h)$ are star-shaped wrt a common point, then $\Omega_1 = \Omega_2$.[4]

[4] Novikoff, P.S.: Sur le problème inverse du potentiel. (1938)

[5] Gustafsson, B., Putinar, M.: Selected topics on quadrature domains. (2007)

[6] Aharonov, D., Shapiro, H.S.: Domains on which analytic functions satisfy quadrature identities. (1976)

The Inverse and Direct Problems for LQDs

Question: When is an LQD uniquely associated to its quadrature function? How can we recover it?

These are open problems even for *classical* QDs.

Some known results for QDs:

- The disk is the unique finitely connected *bounded* domain with the MVP.
- $\Omega \in QD(0)$ (Ω is a "null" QD) iff Ω is an exterior disk
- If $\Omega_1, \Omega_2 \in QD(h)$ are star-shaped wrt a common point, then $\Omega_1 = \Omega_2$.[4]
- If a s.c. bounded domain $\Omega \in QD(h)$ for some $h(w) = \sum_{j=1}^{n} \frac{c_j}{w - w_j}$ with the $c_j > 0$ and $\operatorname{diam}(\{w_j\}) < \sqrt{c_1 + \ldots + c_n}$, then Ω is unique.[5]

---

[4] Novikoff, P.S.: Sur le problème inverse du potentiel. (1938)
[5] Gustafsson, B., Putinar, M.: Selected topics on quadrature domains. (2007)
[6] Aharonov, D., Shapiro, H.S.: Domains on which analytic functions satisfy quadrature identities. (1976)

The Inverse and Direct Problems for LQDs

Question: When is an LQD uniquely associated to its quadrature function? How can we recover it?

These are open problems even for *classical* QDs.

Some known results for QDs:

- The disk is the unique finitely connected *bounded* domain with the MVP.
- $\Omega \in \mathrm{QD}(0)$ (Ω is a "null" QD) iff Ω is an exterior disk
- If $\Omega_1, \Omega_2 \in \mathrm{QD}(h)$ are star-shaped wrt a common point, then $\Omega_1 = \Omega_2$.[4]
- If a s.c. bounded domain $\Omega \in \mathrm{QD}(h)$ for some $h(w) = \sum_{j=1}^{n} \frac{c_j}{w - w_j}$ with the $c_j > 0$ and $\mathrm{diam}\left(\{w_j\}\right) < \sqrt{c_1 + \ldots + c_n}$, then Ω is unique.[5]
- If a s.c. bounded domain $\Omega \in \mathrm{QD}\left(\frac{c_1}{w - w_0} + \frac{c_2}{(w - w_0)^2}\right)$ for some $c_1 > 0$ and $c_2 \in \mathbb{C}$, then Ω is unique.[6]

[4] Novikoff, P.S.: Sur le problème inverse du potentiel. (1938)
[5] Gustafsson, B., Putinar, M.: Selected topics on quadrature domains. (2007)
[6] Aharonov, D., Shapiro, H.S.: Domains on which analytic functions satisfy quadrature identities. (1976)

Uniqueness of Null LQDs

A domain Ω is referred to as a *null LQD* if $\Omega \in \mathrm{QD}_0(0)$. We prove the following classification theorem for null LQDs.

Theorem 2.8

A domain is a null LQD if and only if it is a disk or exterior disk centered at 0.

Uniqueness of Null LQDs

A domain Ω is referred to as a *null LQD* if $\Omega \in \mathrm{QD}_0(0)$. We prove the following classification theorem for null LQDs.

Theorem 2.8

A domain is a null LQD if and only if it is a disk or exterior disk centered at 0.

Recall that a domain is a null QD iff it is an exterior disk.

Uniqueness of Null LQDs

A domain Ω is referred to as a *null LQD* if $\Omega \in \mathrm{QD}_0(0)$. We prove the following classification theorem for null LQDs.

Theorem 2.8

A domain is a null LQD if and only if it is a disk or exterior disk centered at 0.

Recall that a domain is a null QD iff it is an exterior disk.

Question: Why are disks null LQDs but not null classical QDs?

Uniqueness of Null LQDs

A domain Ω is referred to as a *null LQD* if $\Omega \in \mathrm{QD}_0(0)$. We prove the following classification theorem for null LQDs.

Theorem 2.8

A domain is a null LQD if and only if it is a disk or exterior disk centered at 0.

Recall that a domain is a null QD iff it is an exterior disk.

Question: Why are disks null LQDs but not null classical QDs?

Answer: If $\Omega \in \mathrm{QD}_0(0)$, then Ω must contain either 0 or ∞ because otherwise $1 \in L^1_a(\Omega; \rho_0)$, so

$$0 < \int_\Omega |w|^{-2} dA(w) = \oint_{\partial\Omega} 1 \cdot 0 dw = 0.$$

Whereas, for classical QDs $\Omega \in \mathrm{QD}_0(0)$ implies only $\infty \in \Omega$.

The Faber Transform Method For LQDs

Theorem 2.9

Let $\Omega \in QD_0(h)$ be a bounded simply connected domain with Riemann map φ. Then there exists rational $r \in \mathcal{A}_0(\mathbb{D}_*)$ such that[7]

$$
\begin{aligned}
\varphi(z) &= w_0 e^{r^{\#}(z)}, && \text{when } 0 \notin \Omega, \\
\varphi(z) &= \frac{w_0}{|z_0|} b_{z_0}(z) e^{r^{\#}(z)}, && \text{when } 0 \in \Omega,
\end{aligned}
\tag{1}
$$

Where $\varphi(0) = w_0$ and $\varphi(z_0) = 0$.

[7] $r^{\#}(z) := \overline{r(\overline{z^{-1}})}$.

The Faber Transform Method For LQDs

Theorem 2.9

Let $\Omega \in QD_0(h)$ be a bounded simply connected domain with Riemann map φ. Then there exists rational $r \in \mathcal{A}_0(\mathbb{D}_*)$ such that[7]

$$\begin{aligned} \varphi(z) &= w_0 e^{r^{\#}(z)}, & \text{when } 0 \notin \Omega, \\ \varphi(z) &= \frac{w_0}{|z_0|} b_{z_0}(z) e^{r^{\#}(z)}, & \text{when } 0 \in \Omega, \end{aligned} \tag{1}$$

Where $\varphi(0) = w_0$ and $\varphi(z_0) = 0$. In this case,

$$r(z) = \Phi_{\varphi}^{-1}\left(wh(w) + \operatorname*{Res}_{\infty} h\right)(z), \tag{2}$$

and

$$h(w) = \frac{\Phi_{\varphi}\left(r\right)\left(w\right) + C}{w} \tag{3}$$

[7] $r^{\#}(z) := \overline{r(\overline{z^{-1}})}$.

Example: Non-Singular Bounded One Point LQDs

Theorem 2.10

Fix $\alpha > 0$ and $w_0 \in \mathbb{C} \setminus \{0\}$. If $0 \notin \Omega$ is simply connected and bounded, then $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ iff $0 < \alpha \leq \pi^2$ and $\Omega = \varphi(\mathbb{D})$, where $\varphi(z) = w_0 e^{z\sqrt{\alpha}}$.

Example: Non-Singular Bounded One Point LQDs

Theorem 2.10

Fix $\alpha > 0$ and $w_0 \in \mathbb{C} \setminus \{0\}$. If $0 \notin \Omega$ is simply connected and bounded, then $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ iff $0 < \alpha \leq \pi^2$ and $\Omega = \varphi(\mathbb{D})$, where $\varphi(z) = w_0 e^{z\sqrt{\alpha}}$.

$\Longrightarrow$ **"Proof":** Suppose $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-w_0}\right)$ with Riemann map φ s.t. $\varphi(0) = w_0$. By the theorem,

$$\varphi(z) = w_0 e^{r^{\#}(z)},$$

where

$$r(z) = \alpha w_0 \Phi_{\varphi}^{-1}\left(\frac{1}{w - w_0}\right)(z) = \alpha w_0 \frac{\psi'(w_0)}{z - \psi(w_0)} = \frac{\alpha w_0}{\varphi'(0)} \frac{1}{z}$$

Example: Non-Singular Bounded One Point LQDs

Theorem 2.10

Fix $\alpha > 0$ and $w_0 \in \mathbb{C} \setminus \{0\}$. If $0 \notin \Omega$ is simply connected and bounded, then $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ iff $0 < \alpha \leq \pi^2$ and $\Omega = \varphi(\mathbb{D})$, where $\varphi(z) = w_0 e^{z\sqrt{\alpha}}$.

$\implies$ **"Proof":** Suppose $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-w_0}\right)$ with Riemann map φ s.t. $\varphi(0) = w_0$. By the theorem,

$$\varphi(z) = w_0 e^{r^{\#}(z)},$$

where

$$r(z) = \alpha w_0 \Phi_{\varphi}^{-1}\left(\frac{1}{w - w_0}\right)(z) = \alpha w_0 \frac{\psi'(w_0)}{z - \psi(w_0)} = \frac{\alpha w_0}{\varphi'(0)} \frac{1}{z}$$

Hence, $\varphi(z) = w_0 e^{\frac{\alpha w_0}{\varphi'(0)} z}$. Solving for $\varphi'(0)$, we recover

$$\varphi(z) = w_0 e^{z\sqrt{\alpha}},$$

which is univalent in $\mathbb{D}$ iff $0 < \alpha \leq \pi^2$.

Example: Non-Singular Bounded One Point LQDs

Theorem 2.11

Fix $\alpha > 0$ and $w_0 \in \mathbb{C} \setminus \{0\}$. If $0 \notin \Omega$ is simply connected and bounded, then $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ iff $0 < \alpha \leq \pi^2$ and $\Omega = \varphi(\mathbb{D})$, where $\varphi(z) = w_0 e^{z\sqrt{\alpha}}$.

$\Longleftarrow$ **"Proof":** Suppose $\Omega = \varphi(\mathbb{D})$, where $\varphi(z) = w_0 e^{z\sqrt{\alpha}}$. $\ln(\varphi_{\text{out}})$ is rational, so $\Omega \in \mathrm{QD}_0(h)$.

Example: Non-Singular Bounded One Point LQDs

Theorem 2.11

Fix $\alpha > 0$ and $w_0 \in \mathbb{C} \setminus \{0\}$. If $0 \notin \Omega$ is simply connected and bounded, then $\Omega \in QD_0\left(\frac{\alpha}{w - w_0}\right)$ iff $0 < \alpha \leq \pi^2$ and $\Omega = \varphi(\mathbb{D})$, where $\varphi(z) = w_0 e^{z\sqrt{\alpha}}$.

$\Longleftarrow$ **"Proof":** Suppose $\Omega = \varphi(\mathbb{D})$, where $\varphi(z) = w_0 e^{z\sqrt{\alpha}}$. $\ln(\varphi_{\text{out}})$ is rational, so $\Omega \in \mathrm{QD}_0(h)$.

Then, by the theorem,

$$h(w) - \frac{C}{w} = \frac{\sqrt{\alpha}}{w}\Phi_\varphi\left(z^{-1}\right)(w) = \frac{\sqrt{\alpha}}{w}\frac{\varphi'(0)}{w - \varphi(0)} = \frac{\alpha}{w}\frac{w_0}{w - w_0}$$

Example: Non-Singular Bounded One Point LQDs

Theorem 2.11

Fix $\alpha > 0$ and $w_0 \in \mathbb{C} \setminus \{0\}$. If $0 \notin \Omega$ is simply connected and bounded, then $\Omega \in QD_0\left(\frac{\alpha}{w - w_0}\right)$ iff $0 < \alpha \leq \pi^2$ and $\Omega = \varphi(\mathbb{D})$, where $\varphi(z) = w_0 e^{z\sqrt{\alpha}}$.

$\Longleftarrow$ **"Proof":** Suppose $\Omega = \varphi(\mathbb{D})$, where $\varphi(z) = w_0 e^{z\sqrt{\alpha}}$. $\ln(\varphi_{\mathrm{out}})$ is rational, so $\Omega \in \mathrm{QD}_0(h)$.
Then, by the theorem,

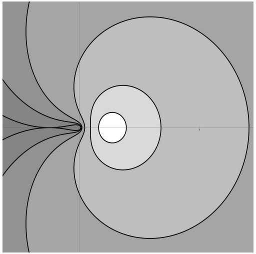
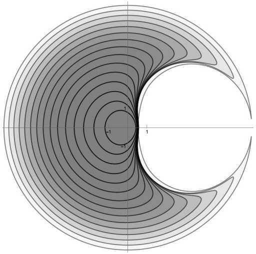
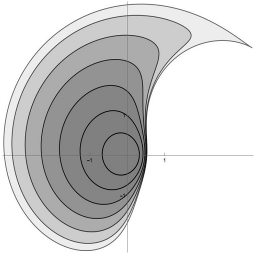
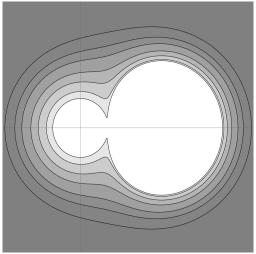
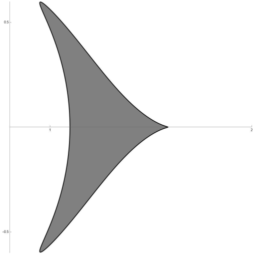
$$h(w) - \frac{C}{w} = \frac{\sqrt{\alpha}}{w}\Phi_\varphi\left(z^{-1}\right)(w) = \frac{\sqrt{\alpha}}{w}\frac{\varphi'(0)}{w - \varphi(0)} = \frac{\alpha}{w}\frac{w_0}{w - w_0}$$

Applying partial fraction decomposition, we obtain

$$h(w) = \frac{\alpha}{w - w_0} + \frac{C - \alpha}{w}.$$

Given that $0 \notin \Omega$, we find that $C - \alpha = 0$, and the conclusion follows.

Classification of One Point LQDs

	Ω **Bounded**	Ω **Unbounded**
$0 \notin \Omega$		 
$0 \in \Omega$		

Non-Singular Unbounded One Point LQDs

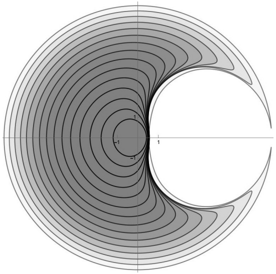
Theorem 2.12

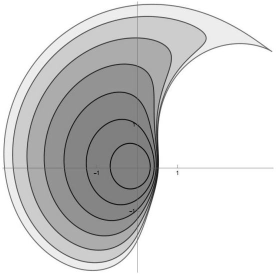
Take $w_0, \alpha \in \mathbb{C} \setminus \{0\}$. There exists an unbounded simply connected domain Ω not containing zero for which $\Omega \in QD_0\left(\frac{\alpha}{w - w_0}\right)$ if and only if there exist $c > 0$, $\lambda \in \mathbb{C}$, and $z_1 \in \mathbb{D}_*$ for which $\Omega = \varphi(\mathbb{D}_*)$, where φ is univalent and given by

$$\varphi(z) = cze^{\frac{\lambda}{1 - z\overline{z_1}}}, \tag{4}$$

where $\varphi(z_1) = w_0$, and $\lambda = \frac{\overline{\alpha w_0}}{\overline{z_1 \varphi'(z_1)}}$.

$w_0 = 1.9$
$\alpha = 1.5, 1.5 + .3i$





Singular Bounded One Point LQDs

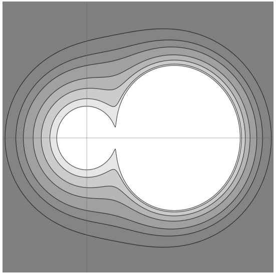
Theorem 2.13

Take $w_0, \alpha \in \mathbb{C} \setminus \{0\}$. There exists a bounded simply connected domain Ω containing zero for which $\Omega \in QD_0\left(\frac{\alpha}{w - w_0}\right)$ if and only if there exist $\lambda \in \mathbb{C}$ and $z_0 \in \mathbb{D}$ for which $\Omega = \varphi(\mathbb{D})$, where φ is univalent and given by

$$\varphi(z) = \frac{w_0}{|z_0|} b_{z_0}(z) e^{\lambda z}, \tag{5}$$

where $\lambda = \frac{\overline{\alpha}}{\varphi'(0)}$.

$w_0 = 1$
$\alpha = .7$



Singular Unbounded One Point LQDs

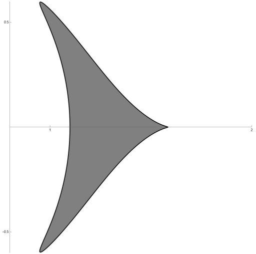
Theorem 2.14

Take $w_0, \alpha \in \mathbb{C} \setminus \{0\}$. There exists an unbounded simply connected domain Ω containing zero for which $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ if and only if there exist $\lambda \in \mathbb{C}$ and $z_0, z_1 \in \mathbb{D}_*$, for which $\Omega = \varphi(\mathbb{D}_*)$, where φ is univalent and given by

$$\varphi(z) = c|z_0| z b_{z_0}(z) e^{\frac{\lambda}{1-z\overline{z_1}}}, \tag{6}$$

where $\varphi(z_1) = w_0$ and $\lambda = \frac{\overline{\alpha w_0}}{\overline{z_1 \varphi'(z_1)}}$.

$w_0 = 2,$
$\alpha = -.15$



Caltech

Future work

- Generalize beyond simply connected domains.

Future work

- Generalize beyond simply connected domains.
    - There are generalizations of Faber transform to multiply connected domains, but not as straightforward to work with.

Future work

- Generalize beyond simply connected domains.
  - There are generalizations of Faber transform to multiply connected domains, but not as straightforward to work with.
  - Schottky-Klein Prime functions.

Future work

- Generalize beyond simply connected domains.
  - There are generalizations of Faber transform to multiply connected domains, but not as straightforward to work with.
  - Schottky-Klein Prime functions.
- Analyze case in which $\partial\Omega$ contains a singular point.

Future work

- Generalize beyond simply connected domains.
  - There are generalizations of Faber transform to multiply connected domains, but not as straightforward to work with.
  - Schottky-Klein Prime functions.
- Analyze case in which $\partial\Omega$ contains a singular point.
- Generalize fact that the boundary of a QD is an algebraic curve to LQDs.

Future work

- Generalize beyond simply connected domains.
  - There are generalizations of Faber transform to multiply connected domains, but not as straightforward to work with.
  - Schottky-Klein Prime functions.
- Analyze case in which $\partial\Omega$ contains a singular point.
- Generalize fact that the boundary of a QD is an algebraic curve to LQDs.
- Consider quadrature domains with “Abelian” quadrature functions.

Future work

- Generalize beyond simply connected domains.
  - There are generalizations of Faber transform to multiply connected domains, but not as straightforward to work with.
  - Schottky-Klein Prime functions.
- Analyze case in which $\partial\Omega$ contains a singular point.
- Generalize fact that the boundary of a QD is an algebraic curve to LQDs.
- Consider quadrature domains with "Abelian" quadrature functions.
- Characterize topology of generalized QDs?
  - Classical case: Lee & Makarov (2016)

That’s all!

Thank you!

The Faber Transform Method for Unbounded LQDs

Theorem 3.1

Let $\Omega \in QD_0(h)$ be an unbounded simply connected domain with Riemann map φ. Then there exists $r \in \mathcal{A}(\mathbb{D})$ such that

$$\begin{aligned} \varphi(z) &= cze^{r^{\#}(z)}, && \text{when } 0 \notin \Omega, \\ \varphi(z) &= c|z_0|zb_{z_0}(z)e^{r^{\#}(z)}, && \text{when } 0 \in \Omega, \end{aligned} \tag{7}$$

Where $\varphi(z_0) = 0$.

The Faber Transform Method for Unbounded LQDs

Theorem 3.1

Let $\Omega \in QD_0(h)$ be an unbounded simply connected domain with Riemann map φ. Then there exists $r \in \mathcal{A}(\mathbb{D})$ such that

$$\begin{aligned} \varphi(z) &= cze^{r^{\#}(z)}, && \text{when } 0 \notin \Omega, \\ \varphi(z) &= c|z_0|zb_{z_0}(z)e^{r^{\#}(z)}, && \text{when } 0 \in \Omega, \end{aligned} \tag{7}$$

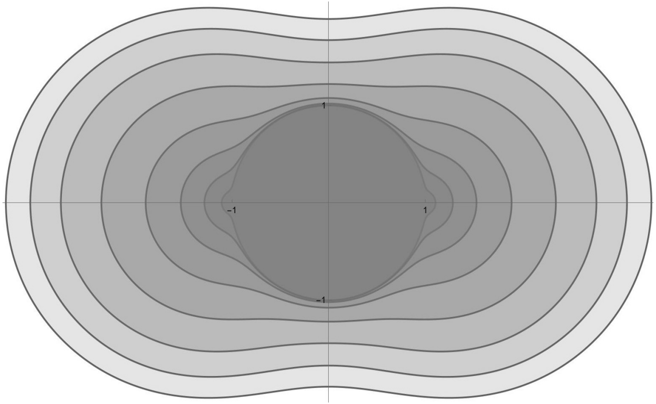
Where $\varphi(z_0) = 0$. In this case,

$$r(z) = \Phi_{\varphi}^{-1}\left(wh(w)\right)(z) - \Phi_{\varphi}^{-1}\left(wh(w)\right)(0). \tag{8}$$

and

$$h(w) = \frac{\Phi_{\varphi}\left(r\right)\left(w\right)}{w} \tag{9}$$

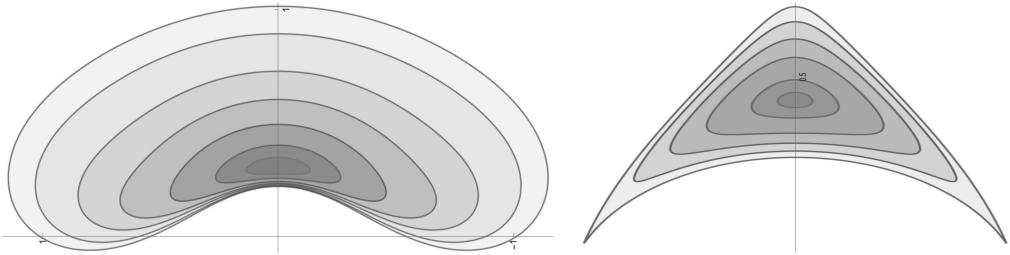
Symmetric Two Point LQDs



Family of LQDs in $QD_0\left(\frac{\alpha}{w-1}+\frac{\alpha}{w+1}\right)$, for $0<\alpha<1$.

$$\varphi(z) = cze^{\frac{2z_+\alpha}{\varphi'(z_+)}\frac{z^2}{1-z^2z_+^2}}$$

Singular Monomial LQDs

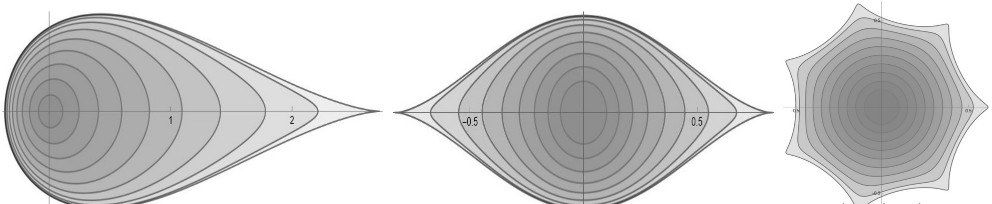


Family of complements of singular monomial LQDs in $QD_0(2\alpha w)$. For $\alpha = -2$, $q = -2$, $0 < c \le .3$ (left); and $\alpha = 1$, $q = -2$, $0 < c < .25$ (right). The plot is rotated so the positive real axis points upwards.

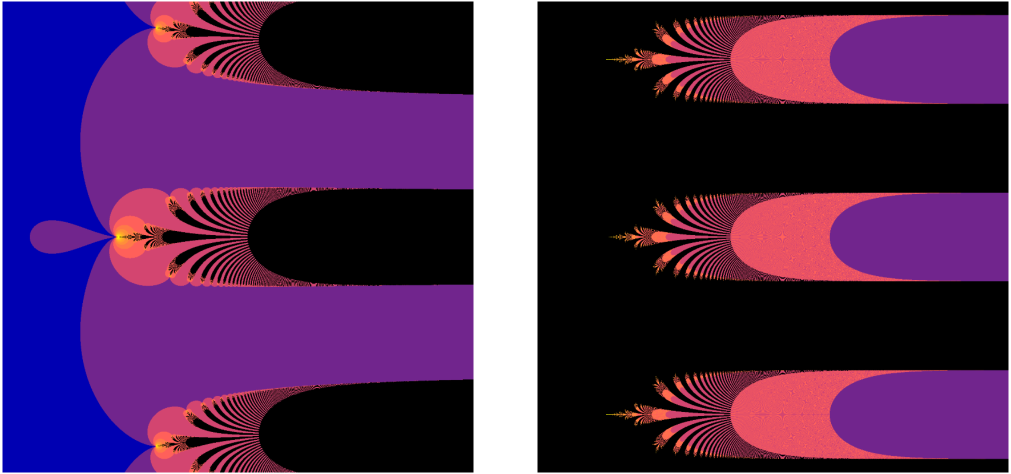
$$\varphi(z) = c|z_0|zb_{z_0}(z)e^{2\overline{\alpha}\left(c^2z^{-2}+\beta z^{-1}\right)}.$$

Monomial LQDs

If $0 \notin \Omega \in \mathrm{QD}_0(\alpha k w^{k-1})$ is simply connected, then $\varphi(z) = cze^{\overline{\alpha} k c^k z^{-k}}$ is a Riemann map for Ω. (c is the conformal radius of Ω)



Family of domains associated to an unbounded simply connected $\Omega \in \mathrm{QD}_0\left(kw^{k-1}\right)$, $k \in \{1, 2, 7\}$.



Tiling set of the Schwarz reflection σ associated to $\Omega \in QD_0(1)$ (left) and filled Julia set of the anti-holomorphic exponential map $w \mapsto e^{\overline{w}-1}$ (right).

Power-Weighted Quadrature Domains

Definition 3.2 (Power-Weighted Quadrature Domain)

We call a bounded (resp. unbounded) domain $\Omega \subset \widehat{\mathbb{C}}$ a *power-weighted* QD (PQD) of order $n \in \mathbb{Z}_+$ if $\exists h \in \mathrm{Rat}_0(\Omega)$ (resp. $\mathrm{Rat}(\Omega)$) s.t

$$\int_\Omega f d\mu = \frac{1}{2\pi i}\oint_{\partial\Omega} f(w)h(w)dw, \qquad (d\mu(w) = n^2|w|^{2(n-1)}dA(w))$$

$\forall f \in \mathcal{A}(\Omega)$ (resp. $A_0(\Omega)$). Denoted by $\Omega \in \mathrm{QD}_n(h)$.

Power-Weighted Quadrature Domains

Definition 3.2 (Power-Weighted Quadrature Domain)

We call a bounded (resp. unbounded) domain $\Omega \subset \widehat{\mathbb{C}}$ a *power-weighted* QD (PQD) of order $n \in \mathbb{Z}_+$ if $\exists h \in \mathrm{Rat}_0(\Omega)$ (resp. $\mathrm{Rat}(\Omega)$) s.t

$$\int_\Omega f d\mu = \frac{1}{2\pi i}\oint_{\partial\Omega} f(w)h(w)dw, \qquad (d\mu(w) = n^2|w|^{2(n-1)}dA(w))$$

$\forall f \in \mathcal{A}(\Omega)$ (resp. $A_0(\Omega)$). Denoted by $\Omega \in \mathrm{QD}_n(h)$.

Power-Weighted Quadrature Domains

Definition 3.2 (Power-Weighted Quadrature Domain)

We call a bounded (resp. unbounded) domain $\Omega \subset \widehat{\mathbb{C}}$ a *power-weighted* QD (PQD) of order $n \in \mathbb{Z}_+$ if $\exists h \in \mathrm{Rat}_0(\Omega)$ (resp. $\mathrm{Rat}(\Omega)$) s.t

$$\int_\Omega f d\mu = \frac{1}{2\pi i}\oint_{\partial\Omega} f(w)h(w)dw, \qquad (d\mu(w) = n^2|w|^{2(n-1)}dA(w))$$

$\forall f \in \mathcal{A}(\Omega)$ (resp. $A_0(\Omega)$). Denoted by $\Omega \in \mathrm{QD}_n(h)$.

When $n = 1$, we recover classical QDs. On the other hand, when $n > 1$ then the metric is singular at 0. This results in interesting behaviour.

Power-Weighted Quadrature Domains

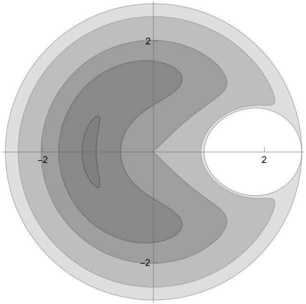
Definition 3.2 (Power-Weighted Quadrature Domain)

We call a bounded (resp. unbounded) domain $\Omega \subset \widehat{\mathbb{C}}$ a *power-weighted* QD (PQD) of order $n \in \mathbb{Z}_+$ if $\exists h \in \mathrm{Rat}_0(\Omega)$ (resp. $\mathrm{Rat}(\Omega)$) s.t

$$\int_\Omega f d\mu = \frac{1}{2\pi i} \oint_{\partial\Omega} f(w)h(w)dw, \qquad (d\mu(w) = n^2|w|^{2(n-1)}dA(w))$$

$\forall f \in \mathcal{A}(\Omega)$ (resp. $A_0(\Omega)$). Denoted by $\Omega \in \mathrm{QD}_n(h)$.

When $n = 1$, we recover classical QDs. On the other hand, when $n > 1$ then the metric is singular at 0. This results in interesting behaviour. For example when $\Omega \in \mathrm{QD}_2\left(\frac{10}{w-2}\right)$, a corner appears when $\partial\Omega$ intersects the origin.



Power-Weighted Faber Transform Identity

The Faber transform formulae for classical quadrature domains generalize nicely to weighted QDs.

Power-Weighted Faber Transform Identity

The Faber transform formulae for classical quadrature domains generalize nicely to weighted QDs.
If $n \in \mathbb{Z}_+$ and $\Omega \in \mathrm{QD}_n(h)$ is bounded and s.c, then

$$\varphi^n(z) = \varphi^n(0) + \Phi_\varphi^{-1}\left(\frac{h(w) + G(w)}{nw^{n-1}}\right)^{\#}(z),$$

for some $G \in \mathcal{A}(\Omega)$.

Power-Weighted Faber Transform Identity

The Faber transform formulae for classical quadrature domains generalize nicely to weighted QDs.
If $n \in \mathbb{Z}_+$ and $\Omega \in \mathrm{QD}_n(h)$ is bounded and s.c, then

$$\varphi^n(z) = \varphi^n(0) + \Phi_\varphi^{-1}\left(\frac{h(w) + G(w)}{nw^{n-1}}\right)^{\#}(z),$$

for some $G \in \mathcal{A}(\Omega)$.

Similarly, if $\Omega \in \mathrm{QD}_n(h)$ is unbounded, then

$$\varphi^n(z) = W_n(z) - W_n(0) + \Phi_\varphi^{-1}\left(\frac{h + G(w)}{R'}\right)^{\#}(z)$$

for some $G \in \mathcal{A}_0(\Omega)$, where $W_n := \Phi_\varphi^{-1}(z^n)$ is the nth "inverse Faber polynomial".

Example

If $0 \notin \Omega \in \mathrm{QD}_n\left(\frac{c}{w-w_0}\right)$ is bounded and s.c. with, $c > 0$, $w_0 \in \mathbb{C} \setminus \{0\}$ with Riemann map φ (wlog taking $\varphi(0) = w_0$, $\varphi'(0) > 0$), then

[8] Similar result in [Dragnev, Legg & Saff (2022)]

Example

If $0 \notin \Omega \in \mathrm{QD}_n\left(\frac{c}{w-w_0}\right)$ is bounded and s.c. with, $c > 0$, $w_0 \in \mathbb{C} \setminus \{0\}$ with Riemann map φ (wlog taking $\varphi(0) = w_0$, $\varphi'(0) > 0$), then

$$\varphi^n(z) = \varphi^n(0) + \Phi_{\varphi}^{-1}\left(\frac{c}{nw^{n-1}(w - w_0)}\right)^{\#}(z)$$

[8] Similar result in [Dragnev, Legg & Saff (2022)]

Example

If $0 \notin \Omega \in \mathrm{QD}_n\left(\frac{c}{w-w_0}\right)$ is bounded and s.c. with, $c > 0$, $w_0 \in \mathbb{C} \setminus \{0\}$ with Riemann map φ (wlog taking $\varphi(0) = w_0$, $\varphi'(0) > 0$), then

$$\begin{aligned} \varphi^n(z) &= \varphi^n(0) + \Phi_\varphi^{-1}\left(\frac{c}{nw^{n-1}(w-w_0)}\right)^{\#}(z) \\ &= w_0^n + \frac{c}{n\overline{w_0}^{n-1}}\Phi_\varphi^{-1}\left(\frac{1}{w-w_0} + O(1)\right)^{\#}(z) \end{aligned}$$

[8] Similar result in [Dragnev, Legg & Saff (2022)]

Example

If $0 \notin \Omega \in \mathrm{QD}_n\left(\frac{c}{w-w_0}\right)$ is bounded and s.c. with, $c > 0$, $w_0 \in \mathbb{C} \setminus \{0\}$ with Riemann map φ (wlog taking $\varphi(0) = w_0$, $\varphi'(0) > 0$), then

$$\begin{aligned}
\varphi^n(z) &= \varphi^n(0) + \Phi_\varphi^{-1}\left(\frac{c}{nw^{n-1}(w-w_0)}\right)^{\#}(z) \\
&= w_0^n + \frac{c}{n\overline{w_0}^{n-1}}\Phi_\varphi^{-1}\left(\frac{1}{w-w_0} + O(1)\right)^{\#}(z) \\
&= w_0^n + \frac{c}{n\overline{w_0}^{n-1}}\frac{\overline{\psi'(w_0)}}{z^{-1} - \overline{\psi(w_0)}} = w_0^n + \alpha z.
\end{aligned}$$

[8] Similar result in [Dragnev, Legg & Saff (2022)]

Example

If $0 \notin \Omega \in QD_n\left(\frac{c}{w-w_0}\right)$ is bounded and s.c. with, $c > 0$, $w_0 \in \mathbb{C} \setminus \{0\}$ with Riemann map φ (wlog taking $\varphi(0) = w_0$, $\varphi'(0) > 0$), then

$$\begin{aligned}\varphi^n(z) &= \varphi^n(0) + \Phi_\varphi^{-1}\left(\frac{c}{nw^{n-1}(w-w_0)}\right)^{\#}(z) \\ &= w_0^n + \frac{c}{n\overline{w_0}^{n-1}}\Phi_\varphi^{-1}\left(\frac{1}{w-w_0} + O(1)\right)^{\#}(z) \\ &= w_0^n + \frac{c}{n\overline{w_0}^{n-1}}\frac{\overline{\psi'(w_0)}}{z^{-1} - \overline{\psi(w_0)}} = w_0^n + \alpha z.\end{aligned}$$

So Ω is an nth root of $\mathbb{D}_\alpha(w_0^n)$.

[8] Similar result in [Dragnev, Legg & Saff (2022)]

Example

If $0 \notin \Omega \in \mathrm{QD}_n\left(\frac{c}{w-w_0}\right)$ is bounded and s.c. with, $c > 0$, $w_0 \in \mathbb{C} \setminus \{0\}$ with Riemann map φ (wlog taking $\varphi(0) = w_0$, $\varphi'(0) > 0$), then

$$\begin{aligned}\varphi^n(z) &= \varphi^n(0) + \Phi_\varphi^{-1}\left(\frac{c}{nw^{n-1}(w-w_0)}\right)^{\#}(z)\\ &= w_0^n + \frac{c}{n\overline{w_0}^{n-1}}\Phi_\varphi^{-1}\left(\frac{1}{w-w_0} + O(1)\right)^{\#}(z)\\ &= w_0^n + \frac{c}{n\overline{w_0}^{n-1}}\frac{\overline{\psi'(w_0)}}{z^{-1} - \overline{\psi(w_0)}} = w_0^n + \alpha z.\end{aligned}$$

So Ω is an nth root of $\mathbb{D}_\alpha(w_0^n)$. Because $1 \in \mathcal{A}(\Omega)$,

$$c = \frac{1}{2\pi i}\oint_{\partial\Omega}\frac{c}{w-w_0}dw = \int_\Omega n^2|w|^{2(n-1)}dA(w) = \alpha^2.$$

So $\alpha = \sqrt{c}$.

[8]Similar result in [Dragnev, Legg & Saff (2022)]

Example

If $0 \notin \Omega \in \mathrm{QD}_n\left(\frac{c}{w-w_0}\right)$ is bounded and s.c. with, $c > 0$, $w_0 \in \mathbb{C} \setminus \{0\}$ with Riemann map φ (wlog taking $\varphi(0) = w_0$, $\varphi'(0) > 0$), then

$$\begin{aligned}\varphi^n(z) &= \varphi^n(0) + \Phi_\varphi^{-1}\left(\frac{c}{nw^{n-1}(w-w_0)}\right)^\#(z)\\ &= w_0^n + \frac{c}{n\overline{w_0}^{n-1}}\Phi_\varphi^{-1}\left(\frac{1}{w-w_0} + O(1)\right)^\#(z)\\ &= w_0^n + \frac{c}{n\overline{w_0}^{n-1}}\,\frac{\overline{\psi'(w_0)}}{z^{-1} - \overline{\psi(w_0)}} = w_0^n + \alpha z.\end{aligned}$$

So Ω is an nth root of $\mathbb{D}_\alpha(w_0^n)$. Because $1 \in \mathcal{A}(\Omega)$,

$$c = \frac{1}{2\pi i}\oint_{\partial\Omega}\frac{c}{w-w_0}\,dw = \int_\Omega n^2|w|^{2(n-1)}\,dA(w) = \alpha^2.$$

So $\alpha = \sqrt{c}$. $\longrightarrow$ $\Omega = \left(\mathbb{D}_{\sqrt{c}}(w_0^n)\right)^{\frac{1}{n}}$ (taking the principal branch of $(\cdot)^{\frac{1}{n}}$).[8]

[8] Similar result in [Dragnev, Legg & Saff (2022)]

Preview: Formulae for Higher Order One Point PQDs

Lemma 3.3

Let $n \in \mathbb{Z}_+$, $\Omega \ni w_0$ be a bounded, simply connected domain admitting a Riemann map $\varphi : \mathbb{D} \to \Omega$, $\varphi(0) = w_0$, and $X_k(w_0)$ the space of rational functions $= 0$ at ∞ with a unique pole of order $k \in \mathbb{Z}_+$ at w_0. Then,

1. *If $0 \notin \Omega$, then there exists $h \in X_k(w_0)$ such that $\Omega \in QD_n(h)$ if and only if there exist $z_1, \ldots, z_k \in \mathbb{D} \setminus \{0\}$ such that*

$$\varphi(z) = w_0 \left(1 - zz_1\right)^{\frac{1}{n}} \ldots \left(1 - zz_k\right)^{\frac{1}{n}}.$$

Lemma 3.3

Let $n \in \mathbb{Z}_+$, $\Omega \ni w_0$ be a bounded, simply connected domain admitting a Riemann map $\varphi : \mathbb{D} \to \Omega$, $\varphi(0) = w_0$, and $X_k(w_0)$ the space of rational functions $= 0$ at ∞ with a unique pole of order $k \in \mathbb{Z}_+$ at w_0. Then,

1. *If $0 \notin \Omega$, then there exists $h \in X_k(w_0)$ such that $\Omega \in QD_n(h)$ if and only if there exist $z_1, \ldots, z_k \in \mathbb{D} \setminus \{0\}$ such that*
$$\varphi(z) = w_0 (1 - zz_1)^{\frac{1}{n}} \ldots (1 - zz_k)^{\frac{1}{n}} .$$
2. *If $w_0 = 0$ (so $0 \in \Omega$), then there exists $h \in X_k(0)$ such that $\Omega \in QD_n(h)$ if and only if there exist $z_1, \ldots, z_{k-1} \in \mathbb{D} \setminus \{0\}$ and $a > 0$ such that*
$$\varphi(z) = az (1 - zz_1)^{\frac{1}{n}} \ldots (1 - zz_{k-1})^{\frac{1}{n}} .$$

Preview: Formulae for Higher Order One Point PQDs

Lemma 3.3

Let $n \in \mathbb{Z}_+$, $\Omega \ni w_0$ be a bounded, simply connected domain admitting a Riemann map $\varphi : \mathbb{D} \to \Omega$, $\varphi(0) = w_0$, and $X_k(w_0)$ the space of rational functions $= 0$ at ∞ with a unique pole of order $k \in \mathbb{Z}_+$ at w_0. Then,

1. *If $0 \notin \Omega$, then there exists $h \in X_k(w_0)$ such that $\Omega \in QD_n(h)$ if and only if there exist $z_1, \ldots, z_k \in \mathbb{D} \setminus \{0\}$ such that*
$$\varphi(z) = w_0 (1 - zz_1)^{\frac{1}{n}} \ldots (1 - zz_k)^{\frac{1}{n}}.$$
2. *If $w_0 = 0$ (so $0 \in \Omega$), then there exists $h \in X_k(0)$ such that $\Omega \in QD_n(h)$ if and only if there exist $z_1, \ldots, z_{k-1} \in \mathbb{D} \setminus \{0\}$ and $a > 0$ such that*
$$\varphi(z) = az (1 - zz_1)^{\frac{1}{n}} \ldots (1 - zz_{k-1})^{\frac{1}{n}}.$$
3. *If $0 \in \Omega$ and $w_0 \neq 0$, then there exists $h \in X_k(w_0)$ such that $\Omega \in QD_n(h)$ if and only if there exist $z_1, \ldots, z_k \in \mathbb{D} \setminus \{0\}$ such that*
$$\varphi(z) = \frac{w_0}{|z_1|^2} \frac{z - z_1}{z - \overline{z_1}^{-1}} (1 - z\overline{z_1})^{\frac{1}{n}} (1 - zz_2)^{\frac{1}{n}} \ldots (1 - zz_k)^{\frac{1}{n}}.$$

Preview: Formulae for Higher Order One Point PQDs

Lemma 3.4

Let $\Omega \ni w_0$ be an unbounded, simply connected domain with conformal radius a admitting a Riemann map $\varphi : \mathbb{D}^- \to \Omega$, and $X_k(w_0)$ the space of rational functions $= 0$ at ∞ with a unique pole of order $k \in \mathbb{Z}_+$ at w_0. Then

1. *If $0 \notin \Omega$, then there exists $h \in X_k(w_0)$ such that $\Omega \in QD_n(h)$ if and only if there exists $z_0 \in \mathbb{D} \setminus \{0\}$ with $\varphi\left(\overline{z_0}^{-1}\right) = w_0$, and $z_1, \ldots, z_k \in \mathbb{D} \setminus \{0, z_0\}$ such that*

$$\varphi(z) = az\left(1 - \frac{z_0}{z}\right)^{-\frac{k}{n}}\left(1 - \frac{z_1}{z}\right)^{\frac{1}{n}} \ldots \left(1 - \frac{z_k}{z}\right)^{\frac{1}{n}}.$$

Preview: Formulae for Higher Order One Point PQDs

Lemma 3.4

Let $\Omega \ni w_0$ be an unbounded, simply connected domain with conformal radius a admitting a Riemann map $\varphi : \mathbb{D}^- \to \Omega$, and $X_k(w_0)$ the space of rational functions $= 0$ at ∞ with a unique pole of order $k \in \mathbb{Z}_+$ at w_0. Then

1. *If $0 \notin \Omega$, then there exists $h \in X_k(w_0)$ such that $\Omega \in QD_n(h)$ if and only if there exists $z_0 \in \mathbb{D} \setminus \{0\}$ with $\varphi\left(\overline{z_0}^{-1}\right) = w_0$, and $z_1, \ldots, z_k \in \mathbb{D} \setminus \{0, z_0\}$ such that*
$$\varphi(z) = az\left(1 - \frac{z_0}{z}\right)^{-\frac{k}{n}}\left(1 - \frac{z_1}{z}\right)^{\frac{1}{n}} \ldots \left(1 - \frac{z_k}{z}\right)^{\frac{1}{n}}.$$

2. *If $0 \in \Omega$ then there exists $h \in X_k(w_0)$ such that $\Omega \in QD_n(h)$ if and only if there exist $z_0, z_1 \in \mathbb{D} \setminus \{0\}$ with $\varphi(\overline{z_0}^{-1}) = w_0$, and $z_2, \ldots, z_k \in \mathbb{D} \setminus \{0, z_0, z_1\}$ such that*
$$\varphi(z) = az\left(1 - \frac{z_0}{z}\right)^{-\frac{k}{n}} \frac{z - \overline{z_1}^{-1}}{z - z_1}\left(1 - \frac{z_1}{z}\right)^{\frac{1}{n}} \ldots \left(1 - \frac{z_k}{z}\right)^{\frac{1}{n}}.$$

Preview: Formulae for Higher Order One Point PQDs

Lemma 3.5

Let Ω be an unbounded, simply connected domain with conformal radius a, admitting a Riemann map $\varphi : \mathbb{D}^- \to \Omega$. Then for each $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}_{\geq 0}$,

1. *If $0 \notin \Omega$, then there exists a polynomial h of degree k such that $\Omega \in QD_n(h)$ if and only if there exist $z_1, \ldots, z_{k+1} \in \mathbb{D} \setminus \{0\}$ such that*

$$\varphi(z) = az\left(1 - \frac{z_1}{z}\right)^{\frac{1}{n}} \ldots \left(1 - \frac{z_{k+1}}{z}\right)^{\frac{1}{n}}.$$

Preview: Formulae for Higher Order One Point PQDs

Lemma 3.5

Let Ω be an unbounded, simply connected domain with conformal radius a, admitting a Riemann map $\varphi : \mathbb{D}^- \to \Omega$. Then for each $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}_{\geq 0}$,

1. *If $0 \notin \Omega$, then there exists a polynomial h of degree k such that $\Omega \in QD_n(h)$ if and only if there exist $z_1, \dots, z_{k+1} \in \mathbb{D} \setminus \{0\}$ such that*
$$\varphi(z) = az\left(1 - \frac{z_1}{z}\right)^{\frac{1}{n}} \dots \left(1 - \frac{z_{k+1}}{z}\right)^{\frac{1}{n}}.$$

2. *If $0 \in \Omega$, then there exists a polynomial h of degree k such that $\Omega \in QD_n(h)$ if and only if there exist $z_1, \dots, z_{k+1} \in \mathbb{D} \setminus \{0\}$ such that*
$$\varphi(z) = az\frac{z - \overline{z_1}^{-1}}{z - z_1}\left(1 - \frac{z_1}{z}\right)^{\frac{1}{n}} \dots \left(1 - \frac{z_{k+1}}{z}\right)^{\frac{1}{n}}.$$